

A mechanical approach to mean field spin models

Giuseppe Genovese ^{*} and Adriano Barra^{* †}

December, 11, 2008

Abstract

Inspired by the bridge pioneered by Guerra among statistical mechanics and analytical mechanics on $1 + 1$ continuous Euclidean space-time, we built a self-consistent method to solve for the thermodynamics of mean-field models, whose order parameters self average. We show the whole procedure by analyzing in full details the simplest test case, namely the Curie-Weiss model. Further we report some applications also to models whose order parameters do not self-average, by using the Sherrington-Kirkpatrick spin glass as a guide.

Introduction

Mean field statistical mechanics of discrete systems is experiencing a massive increasing of interest for several reasons. Born as an infinite dimensional limit of a theoretical background for condensed matter physics, mean field statistical mechanics become immediately appealing for its possibility of being solved (even though this happens exactly for really a few models [14]), still retaining several features of more realistic systems with finite dimensionality.

Furthermore, and maybe nowadays, foremost, its range of applicability is continuously spreading such that, so far, it is one of the key tools for the investigation of several models far away from physics like biological or social networks (see for instance, respectively, [12][13] and [5] [15]): all systems where the mean field nature of the description is not a limitation and whose rigorous or heuristic analysis was, in past decades, unimaginable.

As a consequence the need for methods in statistical mechanics is one of the fundamental enquiries raised to theoretical physicists and mathematicians involved in the field.

In this paper, inspired by a pioneering work of Francesco Guerra [8], we develop an alternative approach to standard statistical mechanics to solve for the thermodynamics of systems whose order parameters self-average.

With the aim of presenting the theory also to readers who may not be experts in statistical mechanics, we apply our scheme to the simplest and most well known Curie-Weiss (CW) model, which we solve in full detail, for the sake of simplicity, linking our procedures with general statistical mechanics models via frequent remarks spread throughout the whole paper.

As the largest interest is paid to complex systems, after the CW, we analyze the Sherrington-Kirkpatrick (SK) model, in the replica symmetric regime, subjected to an external field.

^{*}Dipartimento di Fisica, Sapienza Università di Roma

[†]Dipartimento di Matematica, Università di Bologna

Guerra Action for mean field spin models

Even though we will be interested in observable's behavior once the thermodynamic limit is taken, let us consider a large set of N Ising spins $\sigma_i = \pm 1$, $i \in (1, \dots, N)$. Let us deal with a generic mean-field spin model, described by the Hamiltonian

$$H_N(\sigma) = - \sum_{(i,j)}^N \chi_{ij} \sigma_i \sigma_j, \quad (1)$$

where χ_{ij} is a two body interaction matrix. The main quantity of interest in statistical mechanics is the infinite volume limit of the free energy $f(\beta) = \lim_{N \rightarrow \infty} f_N(\beta) = \lim_{N \rightarrow \infty} -\beta^{-1} A_N(\beta)$, where $A_N(\beta)$ is the pressure and is related to the Hamiltonian via

$$A_N(\beta) = \frac{1}{N} \ln \sum_{\sigma} \exp(-\beta H_N(\sigma)).$$

We stress here (even though we will not deal with disordered systems in the first part of the work) that for the SK model it is usually expected to consider the quenched average of the free energy [8], however, without explicit expectation over the random coupling we mean its value χ -almost surely in the sense of the first Borel-Cantelli lemma.

It is useful to consider the one body interaction, of the same nature of Hamiltonian, that we call *cavity field*

$$H'_N(\sigma) = - \sum_i^N \chi_i \sigma_i.$$

We define further a two parameters *Boltzmannfaktor* $B(x, t)$ and a relative Gibbs measure $\langle \cdot \rangle_{(x,t)}$ as:

$$B_N(x, t) = \exp(\theta(t) H_N + \theta(x) H'_N), \quad (2)$$

$$\langle f(\sigma) \rangle_{(x,t)} = \frac{\sum_{\sigma} f(\sigma) (B(x, t))}{\sum_{\sigma} (B(x, t))}, \quad (3)$$

where θ is a increasing function, vanishing at the origin, strictly dependent by the form of interaction. Eventually a magnetic field can be added in (1), and therefore in (2,3).

We define the Guerra action $\varphi(x, t)$ for a mean field model as the solution of the Hamilton-Jacobi differential equation

$$\partial_t \varphi_N(x, t) + \frac{1}{2} (\partial_x \varphi_N(x, t))^2 + V_N(x, t) = 0, \quad (4)$$

with suitable boundary condition.

Furthermore the function $u(x, t) = \partial_x \varphi(x, t)$ satisfies

$$\partial_t u_N(x, t) + u_N(x, t) \partial_x u_N(x, t) + \partial_x V_N(x, t) = 0. \quad (5)$$

The Guerra action $\varphi_N(x, t)$ is related to the pressure of the model $A_N(\sigma)$, in a way that will be specified later, case by case.

Consequently even the potential function $V_N(x, t)$ expresses thermodynamical quantities of the case study (i.e. in CW and SK models we investigate, it turns out to be the self-averaging of the order parameters).

We will be interested throughout the paper in the region where $V(x, t) = 0$, when we can always solve our equations ¹. In fact these problems are largely studied in literature far away from statistical mechanics [4]. In particular some Theorems, due to Lax [10], are helpful, since under certain hypothesis (that in a nutshell are related to the uniform convexity of the quantity $\frac{1}{2}(\partial_x \varphi(x, t))^2$), give the form of the unique solution of (4) and (5) (related by a derivation). Following Lax we can state the next

Theorem 1. *For a general differential problem*

$$\begin{cases} \partial_t \varphi(x, t) + \frac{1}{2}(\partial_x \varphi(x, t))^2 = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi(x, 0) = h(x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (6)$$

and

$$\begin{cases} \partial_t u(x, t) + u(x, t) \partial_x u(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ u(x, 0) = g(x) & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (7)$$

where $h(x)$ is Lipschitz-continuous, and $g(x) = h'(x) \in \mathcal{L}^\infty$, it does exist and it is unique the function $y(x, t) : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \varphi(x, t) &= \min_y \left\{ \frac{t}{2} \left(\frac{x - y}{t} \right)^2 + h(y) \right\} \\ &= \frac{t}{2} \left(\frac{x - y(x, t)}{t} \right)^2 + h(y(x, t)) \end{aligned} \quad (8)$$

is the unique weak solution of (6), and

$$u(x, t) = \frac{x - y(x, t)}{t} \quad (9)$$

is the unique weak solution of (7). Furthermore, the function $x \rightarrow y(x, t)$ is not-decreasing.

It is worthwhile to remark that the choice of looking for weak solution (that arises naturally in the Lax's theorems) may look as redundant in our case, since we deal with physical quantities (in general smooth functions). Actually it prevents us from the eventual discontinuities of the solutions of (6) and (7). However, a strong solution is a weak solution too and there is no need to change the essence of the Theorem.

Let us start applying this framework to the CW model.

Mean field ferromagnet as a 1-dimensional fluid

The mean field ferromagnetic model is defined by the Hamiltonian

$$H_N(\sigma) = -\frac{1}{N} \sum_{(i,j)}^N \sigma_i \sigma_j + h \sum_i^N \sigma_i.$$

¹We stress however that the formalism we develop can still be applied to general constrained problems ($V(x, t) \neq 0$), even though their resolutions can be prohibitive

It is easily seen that we have resumed in the Hamiltonian both the two body and one body interaction². Thus, choosing $\theta(a) = a$, we can write the (x, t) -dependent *Boltzmannfaktor* as

$$B_N(x, t) = \exp \left(\frac{t}{N} \sum_{(i,j)}^N \sigma_i \sigma_j + x \sum_i^N \sigma_i \right).$$

Remark 1. When dealing with the ferromagnetic Boltzmannfaktor $B_N(x, t)$ above, classical statistical mechanics is recovered of course, in the free field case, by setting $t = \beta$ and $x = 0$.

In the same way the averages $\langle \cdot \rangle_{(x,t)}$ will be denoted by $\langle \cdot \rangle$ whenever evaluated in the sense of statistical mechanics.

A fundamental role is played by the magnetization m which we introduce as

$$m = \lim_{N \rightarrow \infty} m_N = \lim_{N \rightarrow \infty} \sum_i^N \sigma_i, \quad \langle m \rangle = \lim_{N \rightarrow \infty} \frac{\sum_{\sigma} m_N \exp(-\beta H_N(\sigma))}{\sum_{\sigma} \exp(-\beta H_N(\sigma))}.$$

Let us denote $u_N(x, t)$ the 1-dimensional velocity field and $\varphi_N(x, t)$ its dynamic potential (such that $\partial_x \varphi_N(x, t) = u_N(x, t)$). Here the label N reminds us that the analogy is made with the CW model with finite size N (of course we are interested about the thermodynamic limit of the model).

The Guerra action can be written as

$$\varphi_N(x, t) = -\frac{1}{N} \log \sum_{\{\sigma_N\}} \exp \left(\frac{t}{2} N m_N^2 + x N m_N \right) = -A_N(x, t) + O \left(\frac{1}{N} \right), \quad (10)$$

i.e., the mean field CW pressure (up a minus sign) [3], where t stands for the inverse temperature β and x takes into account the external magnetic field h .

Deriving (10) we get

$$u_N(x, t) = -\langle m_N \rangle(x, t), \quad (11)$$

the mean value of the magnetization. So our analogy is now completed, and we can write a fluid equation as a transport equation for $u_N(x, t)$, plus an Hamilton-Jacobi (HJ) equation for $\varphi_N(x, t)$ and a continuity equation, defining the (purely fictitious) density function $\rho(x, t)$.

We notice that the Guerra action $\varphi_N(x, t)$ satisfies an HJ equation where the potential function is the self-averaging of the magnetization. Indeed, since we have

$$\partial_t \varphi_N(x, t) = -\frac{1}{2} \langle m_N^2 \rangle,$$

and

$$\partial_{x^2}^2 \varphi_N(x, t) = \partial_x u_N(x, t) = -\partial_x \langle m_N \rangle(x, t) = -N(\langle m_N^2 \rangle(x, t) - \langle m_N \rangle^2(x, t)),$$

we can easily choose the external pressure for the fluid, that appears as a potential in the HJ equation, as

$$V_N(x, t) = \frac{1}{2} \left(\langle m_N^2 \rangle(x, t) - \langle m_N \rangle^2(x, t) \right), \quad (12)$$

²In ferromagnet the cavity field coincides with the external field

and we have also

$$-\frac{1}{2N}\partial_x^2\varphi_N(x,t) = V_N(x,t). \quad (13)$$

Finally, computing

$$\varphi_N(x,0) = -A_N(x,0) = -\log 2 - \log \cosh x, \quad (14)$$

we can build the differential problem for our hydrodynamical potential $\varphi_N(x,t)$:

$$\begin{cases} \partial_t\varphi_N(x,t) + \frac{1}{2}(\partial_x\varphi_N(x,t))^2 + V_N(x,t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi_N(x,0) = -\log 2 - \log \cosh x & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \quad (15)$$

Remark 2. We stress that by choosing as a boundary a general point on x but $t=0$ (as we did in eq.(14)), we implicitly skipped the evaluation of the two body interaction which is, usually, the hard core of the statistical mechanics calculations as the one body problem trivially factorizes.

Eq. (15) is just the Hamilton-Jacobi equation for a mechanical 1-dimensional system, with time-dependent interactions. We can write it in a more suggestive way, for exalting our hydrodynamical analogy. Indeed, bearing in mind (13), we have

$$\begin{cases} \partial_t\varphi_N(x,t) + \frac{1}{2}(\partial_x\varphi_N(x,t))^2 - \frac{1}{2N}\partial_x^2\varphi_N(x,t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi_N(x,0) = -\log 2 - \log \cosh x & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \quad (16)$$

This equation is more interesting than the first one, for several reasons. At first it is *closed* with respect to the unknown function³. Furthermore it has a clear physical and mathematical meaning: Indeed the presence of a dissipative term suggests the typical viscous fluid behavior, where friction acts against the motion. The *smallness* of this term (that appears with a factor N^{-1}) acts as a mollifier for our differential problem. It may appear even clearer by investigating the equation for $u_N(x,t)$. Deriving with respect to x eq.(16) (and using standard results about for the order of derivation) we obtain

$$\begin{cases} \partial_t u_N(x,t) + u_N(x,t)\partial_x u_N(x,t) - \frac{1}{2N}\partial_x^2 u_N(x,t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ u_N(x,0) = -\tanh(x) & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \quad (17)$$

This is a viscous Burgers' equation, i.e. a very simple Navier-Stokes equation in one dimension. Here the mollifier term is more incisive, since, as we will see soon, when it vanishes (i.e. in thermodynamic limit), it induces the spontaneous \mathbb{Z}_2 symmetry breaking of statistical mechanics by making the solution $u(x,t)$ (i.e. the magnetization) not regular in the whole (x,t) half-plane.

Lastly let us derive the continuity equation that should complete our formal hydrodynamical analogy for the ferromagnetic model. We stress that it does not carry any further information about the model, as it is all contained in (16) and (17)). From the continuity equation we get

$$\begin{aligned} \partial_t \rho_N(x,t) + u_N(x,t)\partial_x \rho_N(x,t) &= -\rho_N(x,t)\partial_x u_N(x,t) \\ &= \rho_N(x,t)2NV_N(x,t). \end{aligned}$$

Writing

$$D_N(x,t) = \partial_t + u_N(x,t)\partial_x = \frac{d}{ds}, \quad (18)$$

³This is actually a feature of the ferromagnets. For instance it is easily seen that it is not trivially closed for the SK pressure because every derivation involves different overlap combination [2].

the differential operator along the stream lines, we obtain the equation for ρ_N

$$D_N(x, t)\rho_N(x, t) = 2NV_N(x, t)\rho_N(x, t), \quad (19)$$

solved by

$$\rho_N(x, t) = \rho_N(0, 0)e^{2N \int ds V(x(s), t(s))} \quad (20)$$

that is

$$\rho_N(x, t) = \frac{1}{2^N} \sum_{\{\sigma\}} \exp [Ntm_N^2 + Nxm_N] = Z_N(2t, x). \quad (21)$$

Resuming, mean field ferromagnets of finite size N is completely equivalent to the 1-dimensional viscous fluid described by equations

$$\begin{cases} \partial_t u_N(x, t) + u_N(x, t)\partial_x u_N(x, t) - \frac{1}{2N}\partial_{x^2}^2 u_N(x, t) = 0 \\ D_N(x, t)\rho_N(x, t) = 2NV_N(x, t)\rho_N(x, t), \end{cases}$$

and in thermodynamic limit, to an Eulerian fluid, such that

$$\begin{cases} \partial_t u(x, t) + u(x, t)\partial_x u(x, t) = 0 \\ \rho(x, t)^{-1}D(x, t)\rho(x, t) = 0. \end{cases}$$

We would like now to link the finite dimensional model with its thermodynamic limit, *i.e.* the viscous fluid with the inviscid one. It is consequently useful to study the free problem

$$\begin{cases} \partial_t \varphi(x, t) + \frac{1}{2}(\partial_x \varphi(x, t))^2 = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi(x, 0) = -\log 2 - \log \cosh x & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (22)$$

and

$$\begin{cases} \partial_t u(x, t) + u(x, t)\partial_x u(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ u(x, 0) = -\tanh x & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \quad (23)$$

With this purpose we can use Theorem 1.

Remark 3. *We stress that via Theorem (1) changing the boundary condition is equivalent to modify the nature of the spin variables in the ferromagnetic model. Since the condition on H is Lipschitz-continuity, such a theorem is valid for every distribution of spin variables with compact support, but not for example for Gaussian ones (at least trivially). We let for future works further investigations [6]. Hereafter anyway we will deal with only dichotomic variables.*

With $h(y) = -\log 2 - \log \cosh y$, $y = x - tu(x, t)$ (given by (9)), we find

$$\varphi(x, t) = \frac{t}{2}u(x, t)^2 - \log 2 - \log \cosh (x - tu(x, t)),$$

and bearing in mind $\varphi = -A$ and $u = -\langle m \rangle$, by setting $t = \beta$ and $x = h$, we gain the usual free energy for mean field ferromagnet

$$f(\beta, h) = -\frac{1}{\beta}A(\beta, h) = \frac{1}{\beta}\varphi(\beta, h) = \frac{1}{\beta} \left\{ \frac{\beta \langle m \rangle^2}{2} - \log \cosh \beta (h + \langle m \rangle) - \log 2 \right\},$$

where of course $\langle m \rangle$ is the limiting value for the magnetization, as we are going to show. We only have to prove convergence for differential problems (16) and (17) to the free ones, respectively (22) and (23). Let us start with the former by stating the following

Theorem 2. *The function*

$$\varphi_N(x, t) = -\frac{1}{N} \log \left[\sqrt{\frac{N}{t}} \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} \exp -N \left(\frac{(x-y)^2}{2t} - \log 2 - \log \cosh y \right) \right] \quad (24)$$

solves the differential problem (6), and it is

$$|\varphi_N(x, t) - \varphi(x, t)| \leq O\left(\frac{1}{N}\right). \quad (25)$$

Proof In order to find a solution of (22), we put⁴

$$\phi_N(x, t) = e^{-N\varphi_N(x, t)}.$$

After a few calculations

$$\begin{aligned} \partial_t \phi_N(x, t) &= \frac{1}{2} N \phi_N(x, t) (\partial_x \varphi_N(x, t))^2 - \frac{1}{2} \phi_N(x, t) \partial_{x^2}^2 \varphi_N(x, t) \\ &= \frac{1}{2N} \partial_{x^2}^2 \phi_N(x, t), \end{aligned} \quad (26)$$

we see that $\phi(x, t)$ solves the heat equation with conductivity $\frac{1}{2N}$ (and a suitable boundary condition):

$$\begin{cases} \partial_t \phi_N(x, t) - \frac{1}{2N} \partial_{x^2}^2 \phi_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \phi_N(x, 0) = 2^{-N} \cosh^{-N} x & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \quad (27)$$

The unique bounded solution of (27) is

$$\phi_N(x, t) = \sqrt{\frac{N}{t}} \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} \exp \left(-N \left(\frac{(x-y)^2}{2t} - \log 2 - \log \cosh y \right) \right)$$

and, bearing in mind $\varphi_N = -\frac{1}{N} \log \phi_N$, we have

$$\varphi_N(x, t) = -\frac{1}{N} \log \left[\sqrt{\frac{N}{t}} \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} \exp \left(-N \left(\frac{(x-y)^2}{2t} - \log 2 - \log \cosh y \right) \right) \right].$$

We notice that, since the uniqueness of the minimum of the function in the exponent (allowed by Theorem (1)), we easily get $\varphi_N \rightarrow \varphi$ when $N \rightarrow \infty$.

Finally bounds on the error can be made via standard techniques. \square

We must now prove an analogue result for the velocity field $u(x, t)$. Since the equations for $\varphi(x, t)$ and $u(x, t)$ are trivially related by a derivation, it is clear that $u_N(x, t) \rightarrow u(x, t)$ uniformly in the thermodynamic limit. Anyway for the sake of completeness (and as a guide for testing other models) we state the following

⁴This is usually known as the Cole-Hopf transform [4].

Theorem 3. *The function*

$$u_N(x, t) = \frac{\int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} \frac{x-y}{t} \exp \left(-N \left(\frac{(x-y)^2}{2t} - \log 2 - \log \cosh y \right) \right)}{\int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} \exp \left(-N \left(\frac{(x-y)^2}{2t} - \log 2 - \log \cosh y \right) \right)} \quad (28)$$

solves the differential problem (23) and it is

$$|u_N(x, t) - u(x, t)| \leq O \left(\frac{1}{\sqrt{N}} \right). \quad (29)$$

Proof The (28) is easily obtained by direct derivation of φ_N in (24).

Again the bound on the error is made via standard techniques. \square

Finally we can state the subsequent

Corollary 1. *It is $V_N(x, t) \leq O(\frac{1}{N})$ a. e..*

Proof For the two previous theorems we have

$$\varphi_N(x, t) = \varphi(x, t) + O\left(\frac{1}{N}\right),$$

thus

$$\partial_t \varphi_N = \partial_t \varphi + O\left(\frac{1}{N}\right)$$

and

$$(\partial_x \varphi_N)^2 = (\partial_x \varphi)^2 + O\left(\frac{1}{N}\right),$$

and therefore, using the Hamilon-Jacobi equation (15) for φ_N , we find

$$\partial_t \varphi + \frac{1}{2}(\partial_x \varphi)^2 + O\left(\frac{1}{N}\right) + V_N = 0,$$

that implies the thesis. \square

What we meant for "a.e." is actually the whole (x, t) positive half-plane, but the line defined by $(x = 0, t > 1)$ as will be well explained in the next section.

Shock waves and spontaneous symmetry breaking

In this section we study more deeply the properties of equation (23). This is an inviscid Burgers' equation, and again we can have a representation of solutions as characteristics [4]. We get

$$u(x, t) = -\tanh(x - u(x, t)t) \quad (30)$$

i.e the well known self consistence relation for the CW model, with trajectories (parameterized by $s \in \mathbb{R}$)

$$\begin{cases} t &= s \\ x &= x_0 - s \tanh x_0. \end{cases} \quad (31)$$

We can immediately state the subsequent

Proposition 1. *In the region of the plane (x, t) , defined by*

$$x \geq -\sqrt{t(t-1)} + \operatorname{arctanh} \left(\sqrt{\frac{t-1}{t}} \right) \quad \text{for } x_0 \geq 0$$

and

$$x \leq -\sqrt{t(t-1)} + \operatorname{arctanh} \left(\sqrt{\frac{t-1}{t}} \right) \quad \text{for } x_0 \leq 0$$

trajectories (31) have no intersection points.

Remark 4. *This last statement defines the onset of ergodicity breaking in the statistical mechanics of the CW model.*

Proof Set for instance $x_0 \geq 0$.

Once fixed $s = \bar{s}$ let us investigate the position at time \bar{s} as a function of the starting point x_0 . We have

$$x(x_0) = x_0 - \bar{s} \tanh x_0.$$

If $x(x_0)$ is monotone with respect to x_0 , then $\forall x_0 \in \mathbb{R} \exists! x(t)$, i.e for every starting point there is an unique position at time t . In other words, two trajectories born in different points of the boundary cannot, at the same time, assume the same position (do not intersect). Hence we have

$$x'(x_0) = 1 - \bar{s}(1 - \tanh^2 x_0) \geq 0 \quad \forall x_0,$$

only if

$$\bar{s} \leq \frac{1}{1 - \tanh^2 x_0},$$

as $1 - \tanh^2 x_0$ always belongs to $[0, 1]$. The last formula implies

$$x_0 \geq \operatorname{arctanh} \left(\sqrt{\frac{t-1}{t}} \right),$$

and bearing in mind the form of trajectories (31) we get

$$x \geq \operatorname{arctanh} \left(\sqrt{\frac{t-1}{t}} \right) - \sqrt{t(t-1)}. \quad (32)$$

The proof is analogue for $x_0 \leq 0$. \square

We must notice that the previous proposition gives the region of the (x, t) plane in which the invertibility of the motion fails. On the other hand, every trajectory has its end point at the intersection with the x -axes, or are all merged in a unique line, that is $(x = 0, t > 1)$.

More rigorously, the curve $(x = 0, t > 1)$ is a discontinuity line for our solution, since it is easily seen that every point of such a line is an intersection point of the trajectories (31). Also we can get by (30) with direct calculation

$$\partial_x u(x, t) = -\frac{1 - u^2}{1 + t(1 - u^2)} < 0, \quad (33)$$

i.e. the velocity field is strictly decreasing along x direction⁵.

Now we name u_+ the limiting value from positive x , and u_- the one from negative x , and state the following

Proposition 2. *It is $0 < u_- = -u_+ < 0$ for a.e. $t > 1$.*

Proof The curve of discontinuity can be parameterized as

$$\begin{cases} t > 1 \\ x = 0, \end{cases}$$

so has zero speed. We have that $\forall t \geq 0$ does exist a neighbors I of $(x = 0)$ such that $u(x, t)$ is smooth on I . Thus, since we know that our $u(x, t)$ is an integral solution, we can use Rankine-Huginiot condition [4] to state

$$u_+^2 = u_-^2.$$

Since for (33) it has to be $u_+ < u_-$ the assert is proven. \square

Remark 5. *We stress that the relation $u_+^2 = u_-^2$, in this context, mirrors the spin-flip symmetry shared by the two minima of the CW model in the broken ergodicity phase, i.e. $|+\langle m \rangle| = |-\langle m \rangle|$.*

It follows that $(x = 0, t > 1)$ is a shock line for the Burgers' equation (23).

On the other hand, of course, $x = 0$ is an equilibrium point for the system, since we have that both position and velocity are zero.

Remark 6. *This property is translated in statistical mechanics to the trivial case of CW model without neither a vanishing external field, such that spontaneous magnetization can never happen.*

We can use it for exploring the well known mechanism of spontaneous symmetry breaking. With this purpose, let us move on a family of straight lines of equation

$$x = \epsilon t - \epsilon.$$

We have infinitely many lines, all converging in $(0, 1)$, that intersect the x -axes in $-\epsilon$. Let us choose for example $\epsilon > 0$, and perform the limit of $u(x, t)$ on the shock line taking the value of $u(x, t)$ by these, and then sending $\epsilon \rightarrow 0$. Since $-\epsilon$ is negative, the intersection point with $t = 0$ is approaching 0 from the left (x^-), meanwhile the limit of u is taken from the right (u_+). In the same way we have that when the intersection point approach to zero from right (x^+), the limiting value of u is taken from left (u_-).

Remark 7. *In our analogy with statistical mechanics one can make the substitution $u(x, t) = -\langle m \rangle(h, \beta)$, $x = h\beta$, $t = \beta$, getting the spontaneous symmetry breaking mechanism, in such a way that $\lim_{h \rightarrow 0^\pm} \langle m \rangle(h, \beta) = m^\pm$.*

⁵This is a particular case of a more general property of the Lax-Oleinik solution [10], named *entropy condition*, that ensures $u(x, t)$ never increases along x . We won't give the general form, that is redundant in this contest, but can be very useful in studying generalized ferromagnet [6].

Conservation laws

We can rewrite the (15) from a mechanical point of view as

$$\partial_t \varphi_N(x, t) + H_N(\partial_x \varphi_N(x, t), x, t) = 0$$

and the Hamiltonian function reads off as ⁶

$$H_N(\partial_x \varphi_N(x, t), x, t) = \frac{p^2(x, t)}{2} + V_N(x, t). \quad (34)$$

Hamilton equations are nothing but characteristics of equation (15):

$$\begin{cases} \dot{x} &= u_N(x, t) \\ \dot{t} &= 1 \\ \dot{p} &= -u_N(x, t) \partial_x u_N(x, t) - \partial_x V_N(x, t) \\ \dot{E} &= -u_N(x, t) \partial_x (\partial_t \varphi_N(x, t)) - \partial_t V_N(x, t), \end{cases} \quad (35)$$

the latter two equations express the conservation laws for momentum and energy for our system, and can be written in form of streaming equations as

$$\begin{cases} D_N u_N(x, t) &= -\partial_x V_N(x, t) \\ D_N (\partial_t \varphi_N(x, t)) &= -\partial_t V_N(x, t). \end{cases}$$

Since in thermodynamic limit the system approaches a free one, bearing in mind that $u_N(x, t) = -\langle m_N \rangle$ and $\partial_t \varphi_N(x, t) = -\frac{1}{2} \langle m_N^2 \rangle$, so $D_N = \partial_t - \langle m_N \rangle \partial_x$, for $N \rightarrow \infty$ we conclude

$$\begin{cases} D_N \langle m_N \rangle &= 0 \\ D_N \langle m_N^2 \rangle &= 0, \end{cases} \quad (36)$$

i.e.

$$\begin{cases} \langle m_N^3 \rangle - 3 \langle m_N \rangle \langle m_N^2 \rangle + 2 \langle m_N \rangle^3 &= O(\frac{1}{N}) \\ (\langle m^4 \rangle - \langle m^2 \rangle^2) - 2 \langle m \rangle \langle m^3 \rangle + 2 \langle m \rangle^2 \langle m^2 \rangle &= O(\frac{1}{N}). \end{cases} \quad (37)$$

We have from Corollary 1 that $\langle m^2 \rangle = \langle m \rangle^2 + O(\frac{1}{N})$ everywhere but on the line $(x = 0, t > 1)$, where anyway $\langle m \rangle = 0$. It is possible to write down a relation that follows from energy conservation: where the potential vanishes, using momentum conservation, giving $\langle m^3 \rangle = \langle m \rangle^3 + O(\frac{1}{N})$, we get

$$\langle m^4 \rangle - \langle m^2 \rangle^2 = O(\frac{1}{N}).$$

Otherwise when the potential is different from zero⁷ we have $\langle m \rangle = 0$, thus the previous formula is still valid, and it holds in all the (x, t) half-plane.

Remark 8. *This is of course a Ghirlanda-Guerra relation [7] for the CW model (i.e. it expresses self-averaging of the internal energy density). As a counterpart, the bare momentum conservation implies the first Aizenman-Contucci [1] relation for $\langle m^3 \rangle$.*

Remark 9. *It is interesting to remark that the orbits of the Nöther groups of the theory coincide with the streaming lines of our fluid, and conservation laws along these lines give well known identities in the statistical mechanics of the model.*

⁶here we name p our velocity u , i.e. the velocity field coincides with the generalized time dependent momentum

⁷Anyway it is a zero measure set.

The Replica Symmetric phase of the Sherrington-Kirkpatrick model

Despite the main goal when dealing with complex systems is a clear scenario of the *Broken Replica Phase*, which, in our languages translates into solving viscous problems as $V_N(x, t) \neq 0$ (and it is posted to future investigations), a detailed analysis of the replica symmetric regime is however immediate within this framework, as pioneered in [8].

The Sherrington-Kirkpatrick Hamiltonian is given by

$$H_N = -\frac{1}{\sqrt{N}} \sum_{(i,j)} J_{ij} \sigma_i \sigma_j + h \sum_i \sigma_i,$$

where J_{ij} are *i.i.d* centered Gaussian variables, with $\mathbb{E}[J_{ij}] = 0$ and $\mathbb{E}[J_{ij}^2] = 1$.

Following [8] we introduce the partition function

$$Z_N(x, t) = \sum_{\{\sigma\}} \exp \left(\sqrt{\frac{t}{N}} \sum_{(i,j)} J_{ij} \sigma_i \sigma_j + \sqrt{x} \sum_i J_i \sigma_i + \beta h \sum_i \sigma_i \right).$$

Accordingly with the normalization factor $1/\sqrt{N}$ of the model, we choose $\theta(a) = \sqrt{a}$; it is important to stress that differently to the ferromagnetic model, the cavity field with strength \sqrt{x} does not coincide with the magnetic field h , that entries in the *Boltzmannfaktor* as an external parameter. Thus our results will hold for every value of h .

The main difference, when introducing thermodynamical quantities (as the free energy) is in an overall average over the random quenched couplings encoded in the interaction matrix. In this sense the averages $\langle . \rangle$ now stand both for the Boltzmann averages (denoted by ω hereafter when dealing with a single set of phase space configuration, $\Omega = \omega \times \omega \times \dots \times \omega$ when dealing with several replicas of the system) and for the averages over the coupling (denoted by \mathbb{E} hereafter), such that $\langle . \rangle = \mathbb{E}\Omega(.)$.

The Guerra action for the SK model reads off as

$$\varphi_N(x, t) = 2A_N - \frac{t}{2} - x. \quad (38)$$

So it has, once introduced the two replica overlap as $q_{12} = N^{-1} \sum_i \sigma_i^{(1)} \sigma_i^{(2)}$,

$$\partial_t \varphi_N = 2\partial_t A_N - \frac{1}{2} = -\frac{1}{2} \langle q_{12}^2 \rangle \quad (39)$$

$$\partial_x \varphi_N = 2\partial_x A_N - 1 = -\langle q_{12} \rangle. \quad (40)$$

Mirroring the mean field ferromagnet, also in this glass model the interaction factorizes at $t = 0$, and, once set $\mathbb{E}_g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dg e^{-\frac{g^2}{2}}$, we have

$$\varphi_N(x, 0) = 2A_N^{SK}(x, 0) - x = 2 \log 2 + 2\mathbb{E}_g \log \cosh(\beta h + g\sqrt{x}) - x.$$

The last formula, together with (39, 40) allows to build the HJ equation for $\varphi_N(x, t)$

$$\begin{cases} \partial_t \varphi_N(x, t) + \frac{1}{2} (\partial_x \varphi_N(x, t))^2 + V_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi_N(x, 0) = 2 \log 2 + 2\mathbb{E}_g \log \cosh(\beta h + g\sqrt{x}) - x & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (41)$$

with

$$V_N(x, t) = \frac{1}{2} \left(\langle q_{12}^2 \rangle - \langle q_{12} \rangle^2 \right). \quad (42)$$

In complete generality, this is an equation more complicated than the ferromagnetic one: Reflecting the complex structure of the RSB phase, the closure of the equation can be obtained only via cumulant expansions of the overlaps in terms of higher order correlation functions [2], *i.e* the potential has no trivial expression in terms of φ_N derivatives. We will study this equation in the Replica Symmetric phase, that is where, in the (x, t, h) domain, $\lim_N V_N = 0$.

The velocity field, accordingly with (40), is

$$u_N(x, t) = -\langle q_{12} \rangle(x, t)$$

and satisfies the transport equation

$$\begin{cases} \partial_t u_N(x, t) + u_N(x, t) \partial_x u_N(x, t) + \partial_x V_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ u_N(x, 0) = -\mathbb{E}_g \tanh^2(\beta h + g\sqrt{x}) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \quad (43)$$

Remark 10. *We stress that naturally in our approach the hyperbolic tangent of the CW model has been mapped into the squared hyperbolic tangent in the SK case, exactly as it happens in statistical mechanics, reflecting the role of the overlap as a proper order parameter with respect to the magnetization.*

Replica symmetry apart, the characteristic trajectories of (43) are not in general straight lines, because of the presence of the potential. We can give an expression for them:

$$\begin{cases} t = s \\ x = x_0 - s \mathbb{E}_g \tanh^2(\beta h + g\sqrt{x_0}) - \int_0^s ds' \partial_x V_N(x(s'), t(s')). \end{cases} \quad (44)$$

and solving for u

$$u_N(x, t) = -\mathbb{E}_g \tanh^2(\beta h + g\sqrt{x_0(x, t)}) - \int ds \partial_x V_N(x(s), s), \quad (45)$$

where we get $x_0(x, t)$ inverting the second among (44).

This is the analogous of the Guerra sum rule for the order parameter q^8 , stating that the difference among the true order parameter and the RS one is the line integral of the x derivative of V_N along trajectories.

Reducing our attention to the RS phase of the model, we get the free HJ equation

$$\begin{cases} \partial_t \varphi_{RS}(x, t) + \frac{1}{2} (\partial_x \varphi_{RS}(x, t))^2 = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ \varphi_{RS}(x, 0) = 2 \log 2 + 2 \mathbb{E}_g \log \cosh(\beta h + g\sqrt{x}) - x & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \quad (46)$$

and Burger's equation

$$\begin{cases} \partial_t u_N(x, t) + u_N(x, t) \partial_x u_N(x, t) = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\ u_N(x, 0) = -\mathbb{E}_g \tanh^2(\beta h + g\sqrt{x}) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \quad (47)$$

⁸Actually Guerra relation may be obtained through an integration of (45).

We are now in perfect agreement with the hypothesis of Theorem (1). Therefore we can write the Replica-Symmetric Guerra action, in the thermodynamic limit, as

$$\varphi_{RS}(x, t) = \frac{t}{2} \left(\frac{x - y(x, t)}{t} \right)^2 + \log 2 + \mathbb{E}_g \log \cosh(\beta h + g\sqrt{y(x, t)}) - y(x, t), \quad (48)$$

and, naming the velocity field of the free problem $-\bar{q}(x, t)$, we trivially get from (45) the self-consistence equation

$$\bar{q}(x, t) = \mathbb{E}_g \tanh^2(\beta h + g\sqrt{x + t\bar{q}(x, t)}), \quad (49)$$

and the trajectories are

$$\begin{cases} t &= s \\ x &= x_0 - s\mathbb{E}_g \tanh^2(\beta h + g\sqrt{x_0}). \end{cases} \quad (50)$$

Remark 11. We stress that eq. (49) is exactly the self-consistent equation for the SK model order parameter in the replica symmetric ansatz.

Furthermore the minimization point $y(x, t)$ is usually given by

$$y(x, t) = x + t\bar{q}(x, t).$$

Proposition 3. For values of t , x and βh such that

$$t\mathbb{E}_g \left[\frac{1}{\cosh^4(\beta h + g\sqrt{x + \bar{q}t})} \right] \leq \frac{1}{3} + \frac{2}{3}t\mathbb{E}_g \left[\frac{1}{\cosh^2(\beta h + g\sqrt{x + \bar{q}t})} \right], \quad (51)$$

trajectories (50) have no intersection points. In particular the whole region with $x \geq 0$ and $t \geq 0$ is included in (51).

Remark 12. We notice that the (51) gives the form of the caustics for the (x, t) motion, i.e.

$$t\mathbb{E}_g \left[\frac{1}{\cosh^4(\beta h + g\sqrt{x + \bar{q}t})} \right] = \frac{1}{3} + \frac{2}{3}t\mathbb{E}_g \left[\frac{1}{\cosh^2(\beta h + g\sqrt{x + \bar{q}t})} \right]$$

and in this sense completes the theorem given in [8].

Proof The procedure is just the same used in Proposition 1. Starting from (50), we put

$$x(x_0) = x_0 - t\mathbb{E}_g \tanh^2(\beta h + g\sqrt{x_0}),$$

i.e. the position depending by initial data, and let's study its monotony. Given the trajectories, it is clear that, whereas there is no intersection, $x(x_0)$ must be increasing, thus

$$\partial_{x_0} x(x_0) = 1 - t\mathbb{E}_g \partial_{x_0} \tanh^2(\beta h + g\sqrt{x_0}) \geq 0,$$

(of course we can swap derivatives and Gaussian integral). So we have

$$t\mathbb{E}_g \partial_{x_0} \tanh^2(\beta h + g\sqrt{x_0}) \leq 1. \quad (52)$$

Now

$$\begin{aligned}
\mathbb{E}_g \partial_{x_0} \tanh^2(\beta h + g\sqrt{x_0}) &= \frac{1}{\sqrt{x_0}} \mathbb{E}_g \left[g \frac{\tanh(\beta h + g\sqrt{x_0})}{\cosh^2(\beta h + g\sqrt{x_0})} \right] \\
&= \frac{1}{\sqrt{x_0}} \mathbb{E}_g \left[\partial_g \frac{\tanh(\beta h + g\sqrt{x_0})}{\cosh^2(\beta h + g\sqrt{x_0})} \right] \\
&= \mathbb{E}_g \left[\frac{1}{\cosh^4(\beta h + g\sqrt{x_0})} \right] - 2 \mathbb{E}_g \left[\frac{\tanh^2(\beta h + g\sqrt{x_0})}{\cosh^2(\beta h + g\sqrt{x_0})} \right] \\
&= 3 \mathbb{E}_g \left[\frac{1}{\cosh^4(\beta h + g\sqrt{x_0})} \right] - 2 \mathbb{E}_g \left[\frac{1}{\cosh^2(\beta h + g\sqrt{x_0})} \right]
\end{aligned}$$

where we have used the well known formula for Gaussian expectation $\mathbb{E}_g [gF(g)] = \mathbb{E}_g [\partial_g F(g)]$. At this point, putting the last expression in (52) we gain the (51). \square

We can finally give the form of the Sherrington-Kirkpatrick solution for the pressure of the model [9][11]. It is

$$\begin{aligned}
A_{RS}(\beta) &= A_{RS}(0, \beta^2) \\
&= \frac{1}{2} \varphi_{RS}(0, \beta^2) + \frac{\beta^2}{4} \\
&= \log 2 + \mathbb{E}_g \log \cosh(\beta h + g\beta\sqrt{q}) + \frac{\beta^2}{4} (1 - \bar{q})^2.
\end{aligned} \tag{53}$$

Conservation laws

In the same way we did for the CW model, we can get relation among overlap from momentum and energy conservation laws, holding in RS regime. It is remarkable that the vanishing, in thermodynamic limit, of an overlap polynomial is associated to a Nöther streaming of mechanical quantities.

With the aim of deepen this last paragraph, let us stating the following

Lemma 1. *Given $F(\sigma^1 \dots \sigma^s)$ as a smooth, well behaved function of s replicas, we have*

$$D \langle F \rangle = \frac{N}{2} \left\langle F \left[\sum_{a \leq b}^s q_{ab}^2 - s \sum_a^s q_{a,s+1}^2 + \frac{s(s+1)}{2} q_{s+1,s+2}^2 \right] \right\rangle$$

The proof of this lemma works via a long and direct calculation, and we will not report it here [2][8].

We stress that the linearity of D implies all our relations approach zero as $O(1/N)$.

We have, in general, that conservation laws for momentum and energy are given by the streaming equation

$$D_N \langle q_{12} \rangle = -\partial_x V_N(x, t) \tag{54}$$

$$D_N \langle q_{12}^2 \rangle = -2\partial_t V_N(x, t). \tag{55}$$

Of course in RS phase, the right hand term of (54) and (55) vanishes when $N \rightarrow \infty$. Although such an approach does not give any information about the way they vanish in thermodynamic limit (we can always write it is $o(1)$), we can write down our relation without problems, since the presence of D forces them to be at least $O(1/N)$. Explicitly we get

$$\begin{aligned} N \langle q_{12}^3 - 4q_{12}q_{23}^2 + 3q_{12}q_{34}^2 \rangle - N \langle q_{12} \rangle \langle q_{12}^2 - 4q_{12}q_{23} + 3q_{12}q_{34} \rangle &= o(1) \\ N \langle q_{12}^4 - 4q_{12}^2q_{23}^2 + 3q_{12}^2q_{34}^2 \rangle - N \langle q_{12} \rangle \langle q_{12}^3 - 4q_{12}q_{23}^2 + 3q_{12}q_{34}^2 \rangle &= o(1), \end{aligned}$$

i.e. conservation of momentum and energy along the streaming lines of the systems (or along free trajectories (50)) implies that in the RS regime

$$\begin{aligned} \langle q_{12}^3 - 4q_{12}q_{23}^2 + 3q_{12}q_{34}^2 \rangle_{(x,t)} - \langle q_{12} \rangle_{(x,t)} \langle q_{12}^2 - 4q_{12}q_{23} + 3q_{12}q_{34} \rangle_{(x,t)} &\leq O\left(\frac{1}{N}\right) \\ \langle q_{12}^4 - 4q_{12}^2q_{23}^2 + 3q_{12}^2q_{34}^2 \rangle_{(x,t)} - \langle q_{12} \rangle_{(x,t)} \langle q_{12}^3 - 4q_{12}q_{23}^2 + 3q_{12}q_{34}^2 \rangle_{(x,t)} &\leq O\left(\frac{1}{N}\right). \end{aligned}$$

Combining the previous results, we get a third relation

$$\langle q_{12}^4 - 4q_{12}^2q_{23}^2 + 3q_{12}^2q_{34}^2 \rangle_{(x,t)} - \langle q_{12} \rangle^2 \langle q_{12}^2 - 4q_{12}q_{23} + 3q_{12}q_{34} \rangle_{(x,t)} \leq O\left(\frac{1}{N}\right), \quad (56)$$

which, in particular we find physically meaningful, when setting $x = 0$ and $t = \beta^2$, because the replica symmetric assumption on the vanishing of the potential is clearly reflected into the overlap labels in the last identity.

If now we neglect the magnetic field ($h = 0$), as we are in the replica symmetric regime, the gauge symmetry holds such that the SK Hamiltonian is left invariant under the transformation $\sigma \rightarrow \sigma \bar{\sigma}$, $\bar{\sigma}$ being a dichotomic variable out from the N -spin Boltzmann average. Matching [2] and [8] in fact it is straightforward to check that gauging the energy conservation we get (again we stress that it holds only at $h = 0$, and obviously at $t = \beta^2$ e $x = 0$)

$$(1 - \langle q_{12}^2 \rangle) \langle q_{12}^4 - 4q_{12}^2q_{23}^2 + 3q_{12}^2q_{34}^2 \rangle \leq O\left(\frac{1}{N}\right)$$

and consequently

$$\langle q_{12}^4 - 4q_{12}^2q_{23}^2 + 3q_{12}^2q_{34}^2 \rangle \simeq O\left(\frac{1}{N}\right),$$

obtaining the well known relation constraining overlaps [1][2].

Conclusions and outlook

In this work we built a self-consistent method to solve for the thermodynamics of mean field systems, encoded by self-averaging order parameters.

Such a method minimally relies on statistical mechanics, essentially just on the boundary conditions of our partial differential equations, and however, involves just straightforward one-body problems.

Within our approach, that we tested on the Curie-Weiss prototype, we obtained the explicit expression for the free energy as a solution of an Hamilton-Jacobi equation defined on a $1 + 1$

Euclidean space time, whose velocity field obeys a suitably defined Burger's equation in the same space.

The critical line defining ergodicity breaking is obtained as a shock wave for a properly defined Cauchy problem. The behavior of the magnetization, thought of as this velocity field, both in the ergodic and in the broken ergodicity phases have also been obtained rigorously.

As instruments involved in our derivation, we obtained rigorously also the existence of the thermodynamic limit for the free energy and the self-averaging of the order parameter.

Despite the problems in relating conserved quantities and discrete symmetries, in our continuous framework, Noether theory is straightforwardly applicable and gives the well known factorization of the momenta of the order parameter, as expected, being the Curie-Weiss a mean field model.

Furthermore we applied the method even to the replica symmetric phase of the Sherrington-Kirkpatrick model, founding full agreement with Guerra's results and stressing other points as the study of the caustics (which shares some similarities with the AT line) and the study of the symmetries, which turn out to be polynomial identities, typical of complex systems (first of all the Aizenman-Contucci relations).

We emphasize that, actually, we believe the method working for a large range of models (*i.e.* with general interacting variables as spherical spins, etc), several interacting spins as P-spin models, etc...). However, of course, it is still not enlarged to cover the case of not self-averaging order parameters (which is mathematically challenge even with already structured methods).

Furthermore a certain interest should be payed trying to apply the method to finite dimensional problems.

We plan to report soon on these topics and their possible applications.

Acknowledgments

The authors are grateful to Francesco Guerra for a priceless scientific interchange. Further they are pleased to thank Pierluigi Contucci, Sandro Graffi, Isaac Perez Castillo and Renato Lucá for useful discussions.

AB work is supported by the SmartLife Project (Ministry Decree 13/03/2007 n.368). GG work is supported by a Techonogical Voucher *B* via the Physics Department of Parma University.

References

- [1] M. Aizenman, P. Contucci, *On the stability of the quenched state in mean field spin glass models*, J. Stat. Phys. **92**, 765 (1998).
- [2] A. Barra, *Irreducible free energy expansion and overlap locking in mean field spin glasses*, J. Stat. Phys. **123**, (2006).
- [3] A. Barra, *The mean field Ising model trhought interpolating techniques*, J. Stat. Phys. **132**, (2008).
- [4] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics , (1998).
- [5] I. Gallo, A. Barra, P. Contucci, *A minimal model for the imitative behaviour in social decision making: theory and comparison with real data*, Math. Mod. and Meth. in Appl. Sc. **34**, (2008).

- [6] G. Genovese, A. Barra, *in preparation*.
- [7] S. Ghirlanda, F. Guerra, *General properties of overlap distributions in disordered spin systems. Towards Parisi ultrametricity*, J. Phys. A, **31** 9149-9155 (1998).
- [8] F. Guerra, *Sum rules for the free energy in the mean field spin glass model*, in *Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects*, Fields Institute Communications **30**, American Mathematical Society (2001).
- [9] S. Kirkpatrick, D. Sherrington, *Solvable model of a spin-glass*, Phys. Rev. Lett. **35** 1792 (1975).
- [10] P. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, (1973).
- [11] M. Mézard, G. Parisi and M. A. Virasoro, *Spin glass theory and beyond*, World Scientific, Singapore (1987).
- [12] G. Parisi, *A simple model for the immune network*, P.N.A.S. **1**, 429-433, (1990).
- [13] A. S. Perelson, G. Weisbuch, *Immunology for physicists*, Rev. Mod. Phys. **69**, Vol.4, (1997).
- [14] M. Talagrand, *Spin glasses: a challenge for mathematicians. Cavity and mean field models*. Springer Verlag (2003).
- [15] D.J. Watts, S. H. Strogatz, *Collective dynamics of 'small-world' networks*. Nature, **393**, (1998).